PMM U.S.S.R., Vol.49, No.2, pp.265-268, 1985
Printed in Great Britain.

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

THE SOLUTION OF A THERMOELASTICITY PROBLEM FOR INHOMOGENEOUS BODIES IN TERMS OF STRESSES BY THE PERTURBATION METHOD*

P.A. KUNTASHEV and IU.V. NEMIROVSKII

Using the general properties of positive-definiteness of the elasticity theory operator, the convergence in an energy metric is proved for the solution of a thermoelasticity problem for a three-dimensional continuously inhomogeneous isotropic body by the perturbation method.

The convergence of a solution by the perturbation method was proved earlier /1/ for the plane problem of the theory of elasticity of inhomogeneous bodies on the basis of properties for representing harmonic functions in terms of an integral with a weak singularity and the properties of multidimensional singular integrals. A comparison is given in /2/ between the exact solution and a solution constructed by the perturbation method for a special case of the plane problem, and their agreement in the domain of convergence is noted. Another approach that relies on the theory of integral equations is described in /3/, where an analysis is given of three-dimensional boundary value problems of the linear theory of elasticity and thermoelasticity for homogeneous and piecewise-homogeneous media, including existence and uniqueness theorems for the solutions.

1. We consider a linearly elastic inhomogeneous isotropic body occupying a finite domain V with surface A. The body is fastened at a part A_1 of the surface A, where there are no displacements u_i. On the remainder A_2 of the surface A, a surface load $p_i(\cdot)$ acts in conjuction with the volume forces $X_i(\cdot)$ and the temperature field $T(\cdot)$ to cause elastic stresses σ_{ij} and strains ε_{ij}.

Moreover, following the general methods described in /4/, we give a formulation of the thermoelasticity problem and we prove the existence and uniqueness of its solution.

Let $\varepsilon_{ij}^c = \alpha(T - T_0)\delta_{ij}$ denote the thermal strain tensor, where α is the coefficient of thermal expansion, T_0 is the initial temperature in the undeformed state, and δ_{ij} is the Kronecker delta. We assume that the nature of the state of stress and strain and the thermal field enables the linear Cauchy relationships for the strain, and the Duhamel-Neumann relationships

$$\varepsilon_{kk} - \varepsilon_{kk}^c = B_1\sigma_{kk}, \quad e_{ij} - e_{ij}^c = B_2 s_{ij} \tag{1.1}$$

to be used.

Here s_{ij}, e_{ij} are the deviator parts of the stress and strain tensors; summation is over the repeated indices, and the functions B_1, B_2, characterizing the compliance distribution in the elastic body, are expressed in terms of Young's modulus E and Poisson's ratio ν in the form

$$B_1 = E^{-1}(1 - 2\nu) > 0, \quad B_2 = E^{-1}(1 - \nu) > 0.$$

We assume the functions $B_1(\cdot)$ and $B_2(\cdot)$ to be continuous and strictly positive in V. We will assume that besides the external load a tensor σ_{ij}^c characterizing it is also given and satisfies the conditions

$$\sigma^c_{ji,j} - X_i = 0 \quad \text{on} \quad V; \quad \sigma^c_{ij} n_j = p_i \quad \text{on} \quad A_2. \tag{1.2}$$

The comma here denotes partial differentiation with respect to the appropriate coordinate, and n_j are the vector components of the unit external normal to the surface A_2. Relative to the nature of the external load and the thermal field we assume that the appropriate

---

*Prikl. Matem. Mekhan., 49, 2, 344-347, 1985

quantities σ^c_{ij} and ε_{ij}^c belong to the space $L_2(V)$ of nine-dimensional vectors of square-summable functions

$$\|\sigma^c\|^2 = (\sigma^c, \sigma^c) = \int \sigma_{ij}^c \sigma_{ij}^c dV < \infty. \ (\varepsilon^c, \varepsilon^c) < \infty. \tag{1.3}$$

We introduce the scalar product

$$[\bar{\sigma}, \sigma]_B = \int \left(\frac{1}{3} B_1 \bar{\sigma}_{kk} \sigma_{nn} - B_2 \bar{s}_{ij} s_{ij}\right) dV \tag{1.4}$$

for stress tensors from $L_2(V)$.

We denote the norm induced by the scalar product (1.4) by $\| \ \|_B$ and call it "energetic". It can be confirmed that the norm $\| \ \|_B$ is equivalent to the norm of $L_2(V)$ and

$$[\sigma, \sigma]_B \leqslant \beta^2 (\sigma, \sigma), \ (\sigma, \sigma) \leqslant \alpha^2 [\sigma, \sigma]_B \tag{1.5}$$
$$(\alpha^2 = (\min_{(\cdot)\in V} \min (B_1(\cdot), \ B_2(\cdot)))^{-1}, \ \beta^2 = \max_{(\cdot)\in V} \max (B_1(\cdot), \ B_2(\cdot))) \ .$$

We introduce the space D of continuously differentiable tensors in V that satisfy the homogeneous equilibrium equations and the homogeneous force boundary conditions

$$D = \{\sigma_{ij} \mid \sigma_{ij} \in C^1(V); \ \sigma_{ji,\,j} = 0 \ \text{ on } \ V; \ \sigma_{ji} n_j = 0 \ \text{ on } \ A_2\} . \tag{1.6}$$

We call the supplement of the space D in the energy metric the energy space Ψ. By construction $D \subset C^1(V) \subset L_2(V)$. We show that $\Psi \subset L_2(V)$. Indeed, since the energy norm and the norm of $L_2(V)$ are identical in conformity with conditions (1.5), the closure /D/ in either metric agrees and equals Ψ. Then the desired embedding $\Psi \subset L_2(V)$ results from the known /5/ closure property $M_1 \subset M_2 \Rightarrow [M_1] \subset [M_2]$ and the completeness of $L_2(V)$

$$D \subset \Psi \subset L_2(V) . \tag{1.7}$$

We call the generalized solution of the thermoelasticity problem the stress distribution $\sigma = \sigma^c + \tau$ that minimizes the Castigliano functional /6/ in

$$K(\tau^*) = \|\sigma^c - \tau^*\|_B^2 - 2(\varepsilon^c, \tau^*) \to \min_{\tau^* \in \Psi} . \tag{1.8}$$

The necessary condition for this functional to have a minimum is

$$\tau : [\tau, \tau^*]_B = l(\tau^*), \ \ \forall \tau^* \in \Psi \ \ l(\tau^*) = -(\varepsilon^c, \tau^*) - [\sigma^c, \tau^*]_B . \tag{1.9}$$

We will show that here l is a linear functional bounded in Ψ. We have (the supremum is taken for $\|\tau\|_B = 1$)

$$\|l\|_B = \sup | - (\varepsilon^c, \tau) - [\sigma^c, \tau]_B | \leqslant \sup \ |(\varepsilon^c, \tau)| + \sup |[\sigma^c, \tau]_B| \leqslant$$
$$\sup (\|\varepsilon^c\|^2 \|\tau\|^2)^{1/2} + \sup (\|\sigma^c\|_B^2 \|\tau\|_B^2)^{1/2} \leqslant \alpha \|\varepsilon^c\| + \beta \|\sigma^c\| < \infty .$$

According to Riesz's theorem /7/, a unique element τ exists in the Hilbert space Ψ that yields a representation of the linear functional l bounded in Ψ in the form of the scalar product (1.9). The Castigliano functional is reduced by substituting (1.9) into (1.8) to the form $K(\tau^*) = \|\tau - \tau^*\|_B^2 - \|\tau\|_B^2 + \|\sigma^c\|_B^2$. which shows that the element $\tau \in \Psi$ satisfying (1.9) achieves the absolute minimum of the Castigliano minimum.

Therefore, the existence and uniqueness of the generalized solution of the thermoelasticity problem are proved.

2. The practical solution of the thermoelasticity problem for an inhomogeneous body will often be fraught with computational difficulties. In many cases it is convenient to use the perturbation method procedure which reduces the thermoelasticity problem for an inhomogeneous body to a series of thermoelasticity problems for a homogeneous body.

We separate the given inhomogeneous distribution of elastic compliances $B_i(\cdot)$ into the sum of a homogeneous distribution of compliances H_i and a perturbing addition $-tb_i(\cdot)$ such that

$$B_i(\cdot) = H_i - tb_i(\cdot), \ 0 < t < 1, \ 0 < b_i(\cdot) < H_i, \ i = 1, \ 2 . \tag{2.1}$$

This can always be done by taking numbers t, H_i satisfying the conditions

$$1 - (\min_{(\cdot)\in V} B_i(\cdot))(\max_{(\cdot)\in V} B_i(\cdot))^{-1} < t < 1; \max_{(\cdot)\in V} B_i(\cdot) < H_i < (1-t)^{-1} \min_{(\cdot)\in V} B_i(\cdot) .$$

We will seek the generalized solution of the thermoelasticity problem (1.9) in the form of a power series in the parameter t

$$\tau_{ij} = \tau_{ij}^{(0)} + \sum_{k=1}^{\infty} t^k \tau_{ij}^{(k)} . \tag{2.2}$$

Substituting (2.1),(2.2) into (1.9), grouping terms and equating those with identical powers of t to zero, we obtain a series of thermoelasticity problems to determine $\tau^{(k)}$

$$\tau^{(0)}: [\tau^{(0)}, \tau^*]_H = -(\varepsilon^\circ, \tau^*) - [\sigma^\circ, \tau^*]_H, \ \forall \tau^* \in \Psi \tag{2.3}$$

$$\tau^{(1)}: [\tau^{(1)}, \tau^*]_H = l_0(\tau^*), \ \forall \tau^* \in \Psi; \ l_0(\tau^*) = [\sigma^\circ + \tau^\circ, \tau^*]_b \tag{2.4}$$

$$\tau^{(k+1)}: [\tau^{(k+1)}, \tau^*]_H = l_k(\tau^*), \ \forall \tau^* \in \Psi; \ \ l_k(\tau^*) = [\tau^{(k)}, \tau^*]_b . \tag{2.5}$$

We will show that the linear functionals $l_k, k = 1, 2, \ldots$ are bounded in Ψ. To do this we use the following inequality that results from inequality (2.1)

$$[\sigma, \sigma]_b \leqslant [\sigma, \sigma]_H . \tag{2.6}$$

Let $\|\tau^{(k)}\|_H < \infty$ then (the supremum is taken for $\|\tau\|_H = 1$) by using (2.6)

$$\|l_k\|_H = \sup |[\tau^{(k)}, \tau]_b| \leqslant \|\tau^{(k)}\|_b \sup \|\tau\|_b \leqslant \|\tau^{(k)}\|_H \sup \|\tau\|_H = \|\tau^{(k)}\|_H < \infty .$$

A similar estimate can also be obtained for the linear functional l_0. Here $\tau^{(0)}$ is bounded in Ψ as a solution of the thermoelasticity problem (2.3) whose properties were studied in Sec.1. From the boundedness of the functional l_0 it follows from the above-mentioned Riesz theorem that the solution $\tau^{(1)}$ of problem (2.4) is defined uniquely, and later by induction all the $\tau^{(k)}$ are defined uniquely from the recursion sequence of problems (2.5).

It is thereby proved that each component of the series (2.2) for the stresses in an inhomogeneous body is defined uniquely.

We will prove that the series (2.2) converges in the energy metric Ψ. Because of the completeness of Ψ, it is sufficient to verify, for this, that

$$\left\| \sum_{k=m}^{n} t^k \tau^{(k)} \right\|_H^2 \to 0 \quad \text{as} \quad m \to \infty, \ n \to \infty . \tag{2.7}$$

We will first establish certain useful properties of the functions $\tau^{(k)}$. Expanding the expression $[\tau^{(m-1)}, \tau^{(n-1)}]_b$ using (2.5) we obtain the property $[\tau^{(m)}, \tau^{(n-1)}]_H = [\tau^{(m-1)}, \tau^{(n)}]_H$, which when applied p times will yield

$$[\tau^{(m)}, \tau^{(n)}]_H = [\tau^{(m-p)}, \tau^{(n+p)}]_H . \tag{2.8}$$

We replace the components to the right of the large quantities by (2.6) in the inequality $2|[\sigma, \tau]_b| \leqslant [\sigma, \sigma]_b + [\tau, \tau]_b$ that follows from the non-negativity of the expression $\|\sigma - \tau\|_b^2$. We obtain the estimate

$$2|[\sigma, \tau]_b| \leqslant [\sigma, \sigma]_H + [\tau, \tau]_H . \tag{2.9}$$

Furthermore, we take $\tau^* = \tau^{(k-1)}$ in relationship (2.5), and using inequality (2.9), we obtain

$$\|\tau^{(k+1)}\|_H^2 \leqslant \|\tau^{(k)}\|_H^2 . \tag{2.10}$$

The property

$$|[\tau^{(k)}, \tau^{(p)}]_H| \leqslant \|\tau^{(1)}\|_H^2, \ k = 1, 2, \ldots; \ p = 1, 2, \ldots \tag{2.11}$$

that results from the Schwartz inequality and the inequality (2.10), also holds.

To prove property (2.7) of the convergence of series (2.2) of the perturbation method, we construct the following chain of inequalities by using properties (2.8) and (2.11):

$$\left\| \sum_{k=m}^{n} t^k \tau^{(k)} \right\|_H^2 = \sum_{p=2m}^{2n} t^p \sum_{i \geqslant m, \, j \geqslant m, \, i+j=p} [\tau^{(i)}, \tau^{(j)}]_H =$$

$$\sum_{p=2m}^{2n} t^p (p - 2m + 1) [\tau^{(p-1)}, \tau^{(1)}]_H \leqslant$$

$$\sum_{p=2m}^{2n} t^p (p - 2m + 1) |[\tau^{(p-1)}, \tau^{(1)}]_H| \leqslant$$

$$\|\tau^{(1)}\|_H^2 \sum_{p=2m}^{2n} (p - 2m + 1) t^p \leqslant \|\tau^{(1)}\|_H^2 \textstyle\sum_{2m, 2n}, \left(\sum_{n, p} = \sum_{k=n}^{p} k t^k \right).$$

The numerical series $\sum_{1,\infty}$ converges under the condition $0<t<1$ assumed, consequently, the sequence of its partial sums is fundamental. Therefore, the last expression tends to zero as $m\to\infty, n\to\infty$ in the chain of estimates presented.

Summarizing, the solution of the thermoelasticity problem for an inhomogeneous body can be sought by the perturbation method in the form of the series (2.2) that converges in the energy space metric and in theequivalent metric $L_2(V)$. The stresses $\tau^{(k)}$ are here defined uniquely from the recursion sequence of problems (2.3),(2.4),(2.5).

REFERENCES

1. MAZILU P., Sur un probleme plan de la theorie de l'elasticité des milieux hétérogènes. Comptes Rendus, Acad Sci., Ser A. & B, Vol.268, No.14, 1969.
2. LOMAKIN B.A. and SHEININ V.I., On the applicability of the small parameter method to estimate the stresses in inhomogeneous elastic media. Izv. Akad. Nauk SSSE, Mekhan. Tverd. Tela, No.3, 1972.
3. KUPRADZE V.D., GEGELIA T.G., BASHELEISHVILI M.O. and BURCHELADZE T.V., Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity. Nauka, Moscow, 1976.
4. MIKHLIN S.G., The Problem of the Minimum of a Quadratic Functional. Gostekhizdat, Moscow-Leningrad, 1952.
5. KOLMOGOROV A.N. and FOMIN S.V., Elements of the Theory of Functions and Functional Analysis. Nauka, Moscow, 1972.
6. NOWACKI W., Theory of Elasticity. Mir, Moscow, 1975.
7. TRENOGIN V.A., Functional Analysis, nauka, Moscow, 1980.

Translated by M.D.F.

PMM U.S.S.R. Vol.49,No.2,pp.268-272,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

EQUILIBRIUM OF A PRESTRESSED ELASTIC BODY WEAKENED BY A PLANE ELLIPTICAL CRACK*

V.M. ALEKSANDROV and B.V. SOBOL'

The problem of normal pressure loading of the edges of a plane elliptical crack is considered. The crack subjected to the load is in the open state. The medium in which it is located is frist subjected to homogeneous biaxial tension or compression along the plane of the crack. A model of incompressible neo-Hooke material is considered /1/. The problem is reduced to solving a singular integral equation of the first kind. In the case when the intensity of the initial loading is identical in both directions, the problem has an exact solution. If the coefficients of preliminary tension differ slightly, construction of the solution of the problem is possible by an asymptotic method /2/. It is shown that as in the case of equal coefficients /3/**, the initial stress does not alter the order of the singularity of the stress field near the crack edge and only affects the normal stress intensity factor. (**See also: Filippova, L.M. On the opening of a circular crack in a prestressed elastic body. Second All-Union Scientific Conference, "Mixed Problems of the Mechanics of a Deformable Body". Abstracts of Reports /in Russian/, Dnepropetrovsk State Univ.., 1981)

Analogous problems are considered in /4, 5/ for the case of equal prestrain coefficients in a body containing a circular crack. A solution /4/ is constructed for the axisymmetric problem for a layer under different conditions on its faces, and it is shown /5/ that it is possible to use the solution of the problem concerning a crack in an anisotropic material. A solution of the axisymmetric problem is constructed /6/ in the case of radial finite prestrain. An asymptotic solution /7/ is obtained for the spatial contact problem for a prestressed elastic body.

1. Let a crack occupying the domain Ω_1 in planform be located in the plane $z=0$ of an elastic space. Uniform loads $\sigma_x = t_1$ and $\sigma_y = t_2$ act in two mutually perpendicular directions

**Prikl. Matem. Mekhan.*, 49,2,348-352,1985